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STABILITY OF VISCOUS FLOW IN A ROTATING POROUS MEDIUM IN THE FORM OF AN ANNULUS: THE SMALL-GAP PROBLEM

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SUMMARY

The paper deals with the linear stability analysis of laminar flow of a viscous fluid in a rotating porous medium in the form of an annulus bounded by two concentric circular impermeable cylinders. The usual no-slip condition is imposed at both the boundaries. The resulting sixth order boundary value, eigenvalue problem has been solved numerically for the small-gap case by the Runge–Kutta–Gill method, assuming that the marginal state is stationary. The results of computation reveal that the critical Taylor number increases with decreasing permeability of the medium. The problem is found to reduce to the case of ordinary viscous flow in the annulus obtained by Chandrasekhar,[1] when the permeability parameter tends to zero.

KEY WORDS Flow in an Annulus Linear Stability Rotating Porous Medium Critical Taylor Number Runge–Kutta–Gill Method

1. INTRODUCTION

Flows in configurations with cylindrical geometry are of importance in many engineering fields, such as electrical motors, lubrication and heat transfer equipment. The practical utility of such flows depends on a good knowledge of their stability. Ever since the publication of the pioneering work of Taylor,[2] the rotational stability problems have attracted the attention of a large number of research workers. But to the authors' knowledge, most of them deal with pure viscous flows between rotating cylinders. In the present paper we investigate the linear stability of flow in a rotating porous medium in the form of an annulus bounded by two concentric circular impermeable cylinders. The flow through the porous medium is analysed by using the Brinkman[3] model for the small-gap case under the assumption that the marginal state is stationary. The resulting sixth order boundary value, eigenvalue problem has been solved employing the Runge–Kutta–Gill method, making use of an iterative procedure given by Fox[4] for correcting the starting values of the integration process. Numerical computations were carried out on a DEC-1090 system.

2. BASIC FLOW

We consider the viscous flow in a porous matrix in the form of an annulus bounded by two concentric circular impermeable cylinders of radii R_1 and R_2 ($>R_1$) (see Figure 1). The flow is caused by the rotation of the matrix with a constant angular velocity Ω. It is assumed that

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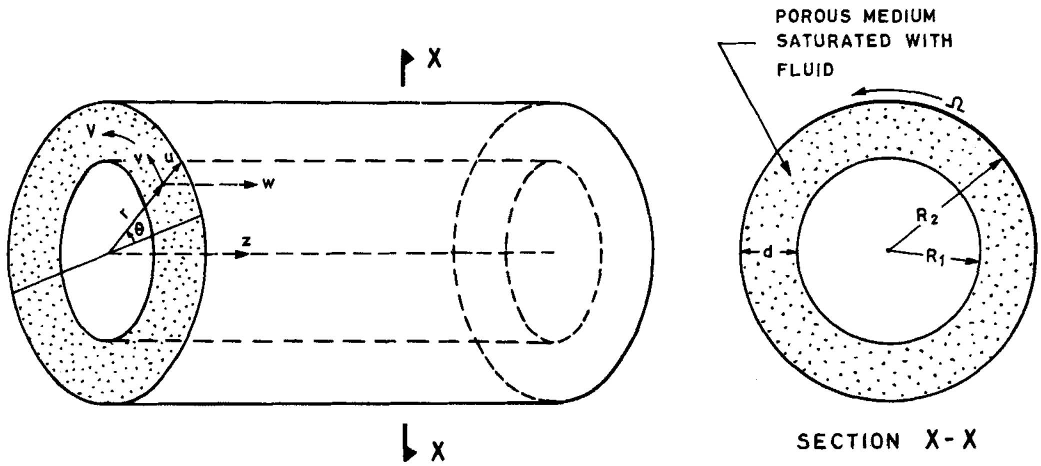

Figure 1. Definition sketch

the porous medium is such that rotation of the annulus will have no effect on its structure and permeability.

The basic flow is obtained by using the Brinkman[3] model:

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{V}{r^2}-\frac{V}{K}=0 \tag{1}$$

$$\frac{dp}{dr}=\frac{\rho V^2}{r} \tag{2}$$

The solution of (1) under the no-slip boundary conditions:

$$V=R_1\Omega \quad \text{at} \quad r=R_1 \tag{3}$$

$$V=R_2\Omega \quad \text{at} \quad r=R_2 \tag{4}$$

is

$$V=A_1I_1\left(\frac{r}{\sqrt{K}}\right)+B_1K_1\left(\frac{r}{\sqrt{K}}\right) \tag{5}$$

where

$$A_1=\frac{\Omega}{N_1}\left[R_1K_1\left(\frac{R_2}{\sqrt{K}}\right)-R_2K_1\left(\frac{R_1}{\sqrt{K}}\right)\right] \tag{6}$$

$$B_1=\frac{\Omega}{N_1}\left[R_2I_1\left(\frac{R_1}{\sqrt{K}}\right)-R_1I_1\left(\frac{R_2}{\sqrt{K}}\right)\right] \tag{7}$$

and

$$N_1=I_1\left(\frac{R_1}{\sqrt{K}}\right)K_1\left(\frac{R_2}{\sqrt{K}}\right)-I_1\left(\frac{R_2}{\sqrt{K}}\right)K_1\left(\frac{R_1}{\sqrt{K}}\right) \tag{8}$$

I_1, K_1 being the modified Bessel functions of order one of the first and second kinds, respectively.

3. STABILITY ANALYSIS

Using the usual normal mode technique[5] we obtain the perturbation equations in the form

$$\frac{\nu}{k^2}\left[\left(DD_*-k^2-\frac{p}{\nu}-\frac{1}{K}\right)(DD_*-k^2)u\right]=2\left(\frac{V}{r}\right)v \tag{9}$$

$$\nu\left[DD_*-k^2-\frac{p}{\nu}-\frac{1}{K}\right]v=(D_*V)u \tag{10}$$

and the continuity equation as

$$D_* u = -kw \tag{11}$$

where

$$D_* = D + \frac{1}{r}, \qquad D = \frac{\mathrm{d}}{\mathrm{d}r}$$

The associated boundary conditions are

$$u = Du = v = 0 \quad \text{at} \quad r = R_1 \text{ and at } r = R_2 \tag{12}$$

From (5)–(8) we get after some simplication,

$$\frac{V}{r} = \frac{\Omega}{N_1[\eta + (1-\eta)\zeta]}[\eta f_1(\zeta) - f_2(\zeta)] \tag{13}$$

$$\left(D + \frac{1}{r}\right)V = \frac{\Omega}{N_1}[f_3(\zeta) + f_4(\zeta)] \tag{14}$$

where

$$\zeta = \frac{r - R_1}{R_2 - R_1} \tag{15}$$

$$\left.\begin{aligned} f_1(\zeta) &= K_1(\sigma)I_1[\sigma(\eta + (1-\eta)\zeta)] - I_1(\sigma)K_1[\sigma(\eta + (1-\eta)\zeta)] \\ f_2(\zeta) &= K_1(\eta\sigma)I_1(\sigma(\eta + (1-\eta)\zeta)] - I_1(\eta\sigma)K_1[\sigma(\eta + (1-\eta)\zeta)] \\ f_3(\zeta) &= [\eta K_1(\sigma) - K_1(\eta\sigma)][\sigma I_0\{\sigma(\eta + (1-\eta)\zeta)\}] \\ f_4(\zeta) &= [I_1(\eta\sigma) - \eta I_1(\sigma)][-\sigma K_0\{\sigma(\eta + (1+\eta)\zeta)\}] \end{aligned}\right\} \tag{16}$$

$$\eta = \frac{R_1}{R_2} \quad \text{and} \quad \sigma = \frac{R_2}{\sqrt{K}}$$

I_0, K_0 being the modified Bessel functions of zero order of the first and second kinds, respectively.

Assuming the thickness of the porous matrix, that is $R_2 - R_1$, to be small compared to the mean radius $(R_1 + R_2)/2$, we approximate D_* by D. Hence the equations (9) and (10) for the marginal stability case (on using (13)–(16)) take the form

$$[D^2 - a^2 - (1-\eta)^2\sigma^2](D^2 - a^2)u = \frac{2a^2d^2\Omega}{\nu N_1\{\eta + (1-\eta)\zeta\}}[\eta f_1(\zeta) - f_2(\zeta)]v \tag{17}$$

and

$$[D^2 - a^2 - (1-\eta)^2\sigma^2]v = \frac{\mathrm{d}^2\Omega}{\nu N_1}[f_3(\zeta) + f_4(\zeta)]u \tag{18}$$

Now using the transformation

$$u = \frac{2a^2d^2\Omega}{\nu}u' \tag{19}$$

in (17) and (18), and writing u in place of u' we get

$$[D^2 - a^2 - (1-\eta)^2\sigma^2](D^2 - a^2)u = \frac{[\eta f_1(\zeta) - f_2(\zeta)]}{N_1[\eta + (1-\eta)\zeta]}v \tag{20}$$

and

$$[D^2 - a^2 - (1-\eta)^2\sigma^2]v = -\frac{Ta^2}{2N_1}[f_3(\zeta) + f_4(\zeta)]u \tag{21}$$

where

$$T = \frac{4\Omega^2 d^4}{\nu^2} \tag{22}$$

$$d = R_2 - R_1 \quad \text{and} \quad a^2 = k^2 d^2 \tag{23}$$

The boundary conditions (12) take the form

$$u = Du = v = 0 \text{ at } \zeta = 0 \text{ and at } \zeta = 1 \tag{24}$$

where now D stands for $d/d\zeta$.

4. NUMERICAL SOLUTION OF THE PROBLEM

In order to obtain the numerical solution of the above problem we use the following transformations:

$$u = Y_1, \quad Du = Y_2, \quad D^2u = Y_3, \quad D^3u = Y_4, \quad v = Y_5, \quad Dv = Y_6 \tag{25}$$

and write the equations (20), (21) and the boundary conditions (24) in the form

$$\left.\begin{aligned}
Y_1' &= Y_2\\
Y_2' &= Y_3\\
Y_3' &= Y_4\\
Y_4' &= [2a^2 + (1-\eta)^2\sigma^2]Y_3 - [a^4 + a^2\sigma^2(1-\eta)^2]Y_1 + \frac{[\eta f_1(\zeta) - f_2(\zeta)]}{N_1[\eta + (1-\eta)\zeta]}Y_5\\
Y_5' &= Y_6\\
Y_6' &= [a^2 + \sigma^2(1-\eta)^2]Y_5 - \frac{Ta^2}{2N_1}[f_3(\zeta) + f_4(\zeta)]Y_1
\end{aligned}\right\} \tag{26}$$

and

$$Y_1 = 0, \quad Y_2 = 0, \quad Y_5 = 0 \quad \text{at} \quad \xi = 0 \text{ and at } \zeta = 1 \tag{27}$$

We solve the problem described by (26) and (27) numerically by employing the Runge–Kutta–Gill method. The integration is to be carried out in [0, 1]. To start the integration it is necessary to know the values of all six functions $Y_1, \ldots, Y_6$ at $\zeta = 0$. But only three of them, namely Y_1, Y_2 and Y_5 are known. So, it is necessary to suitably assume the remaining values Y_3, Y_4 and Y_6. It is also necessary to assume the starting eigenvalue T. If the assumed values (starting values) happen to be correct, the computed values would satisfy the conditions at the second boundary, namely $\zeta = 1$. If this is not the case we have to make corrections in the starting values. For this, we use a self-corrective procedure given by Fox.[4] This procedure involves the integration of three auxiliary initial value problems of the same magnitude as the original problem which again will be solved using the Runge–Kutta–Gill method.

A program was written in Fortran for the Runge–Kutta–Gill method as given by Ralston and Wilf[6] incorporating the corrective procedure. The starting values were computed from the series solution obtained by Chandrasekhar.[1] As a check on the program, the critical Taylor numbers obtained by Chandrasekhar[1] were obtained through the present numerical procedure by taking σ (the permeability parameter) very small. To avoid the vitiation of the results of computation through rounding error, double precision arithmetic was used. The convergence criterion used was that the magnitude of the difference between the computed eigenfunctions (velocity components) and their prescribed values at the outer boundary remained less than or equal to 10^{-8}. It was found that the critical Taylor number obtained underwent a change of about 0·03 per cent with no change in the computed values of the eigenfunctions when the step length of integration was reduced from 0·00125 to 0·000625. So, all integrations were carried out in [0, 1] with the step length equal to 0·000625. The following illustration shows the magnitude of the computer work involved. With the starting values for the eigenfunctions and the eigenvalue T taken from Chandrasekhar's[1] work, the entire process took five iterations to converge (for $\sigma = 10$, $a = 2{\cdot}5$), the CPU time required being 98 s on the DEC-1090 system.

Table I. Results of computation

(a) For $\eta = 0{\cdot}9$ and $\sigma = 6$			
a	$u''(0)$	$u'''(0)$	T_c
1·00	0·02280654	−0·11459771	6462
2·00	0·02109343	−0·11607983	2386
2·50	0·01977162	−0·11556218	1988
3·00	0·01820242	−0·11366216	1859
3·12	0·01779912	−0·11298636	1854
3·25	0·01735402	−0·11216019	1856
3·50	0·01648116	−0·11030869	1885
4·00	0·01471573	−0·10569583	2025
5·00	0·01140578	−0·09414453	2605
6·00	0·00869664	−0·08183139	3617
7·00	0·00664080	−0·07049294	5152
(b) For $\eta = 0{\cdot}9$ and $\sigma = 8$			
1·00	0·02446753	−0·12177304	6936
2·00	0·02317821	−0·12601601	2544
2·50	0·02211211	−0·12746434	2112
2·75	0·02147985	−0·12778709	2014
3·00	0·02079139	−0·12779047	1966
3·12	0·02044394	−0·12767329	1958
3·25	0·02005704	−0·12745858	1959
3·50	0·01928844	−0·12679319	1985
4·00	0·01769699	−0·12454232	2121
5·00	0·01459154	−0·11739002	2699
6·00	0·01193420	−0·10882729	3699
7·00	0·00984285	−0·10065290	5192
(c) For $\eta = 0{\cdot}9$ and $\sigma = 10$			
1·00	0·02736930	−0·13440704	7543
1·75	0·02713518	−0·14147091	3201
2·00	0·02698582	−0·14428807	2742
2·50	0·02655898	−0·15021769	2263
2·75	0·02628387	−0·15322803	2151
3·00	0·02597435	−0·15622752	2093
3·12	0·02581598	−0·15766077	2081
3·15	0·02577558	−0·15801853	2080
3·50	0·02528856	−0·16219029	2098
3·75	0·02493555	−0·16520485	2148
4·00	0·02459344	−0·16831015	2225
5·00	0·02362165	−0·18344679	2782
6·00	0·02407296	−0·21017147	3738
7·00	0·02787159	−0·27048229	5141

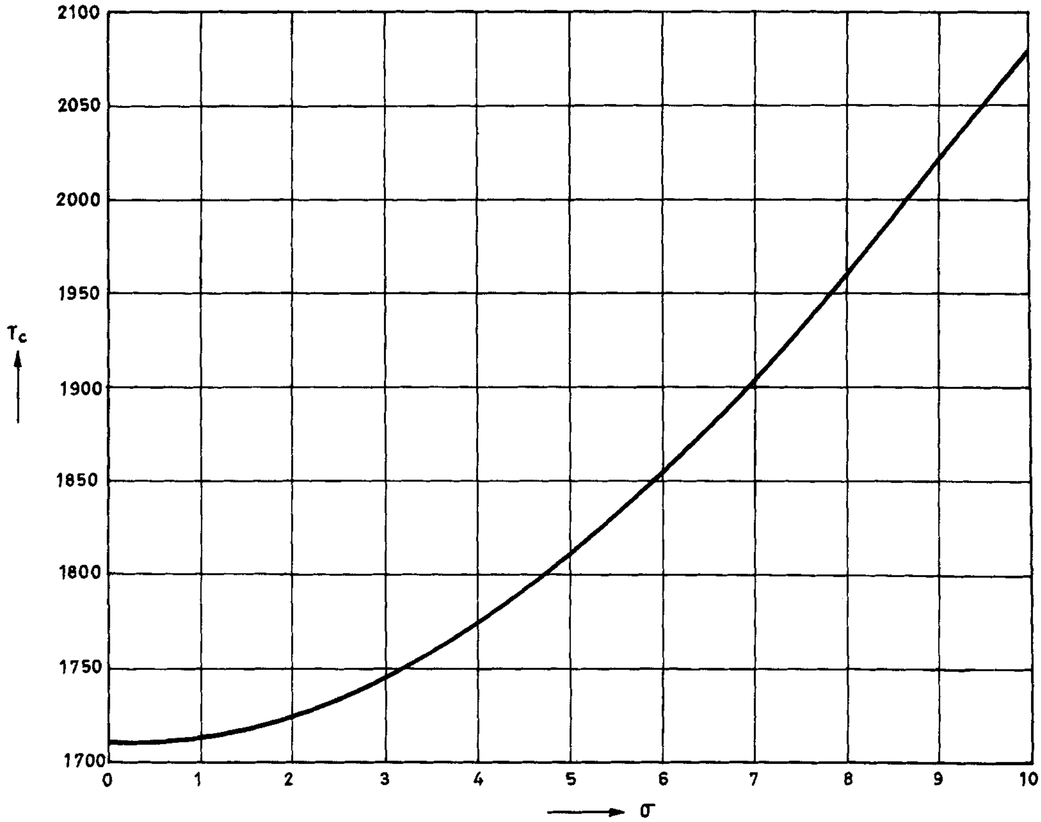

Figure 2. Dependence of T_c on σ (for $a = 3{\cdot}12$ and $\eta = 0{\cdot}9$)

5. DISCUSSION OF RESULTS

The eigenfunctions D^2u, D^3u and the critical Taylor number T_c are computed for some combinations of the parameters σ and a and are given in Table I. The dependence of T_c on σ is presented in Figure 2 which shows that T_c increases with σ. This is physically justified because an increase in σ means a decrease in permeability, which in turn means the presence of more solid particles than liquid particles in the annular region. The effect of this is to increase the resistance to the flow making the system more stable. We also find that the critical Taylor number obtained by Chandrasekhar[1] (for ordinary viscous flow case) is obtained in the limit as σ tends to zero. The marginal stability curves are sketched in Figure 3 for different σ which shows that the critical value of a increases slightly with increasing σ. The eigenfunctions u, v, w presented in Figures 4, 5 and 6 can be seen to be larger than the corresponding ones for the ordinary viscous flow.

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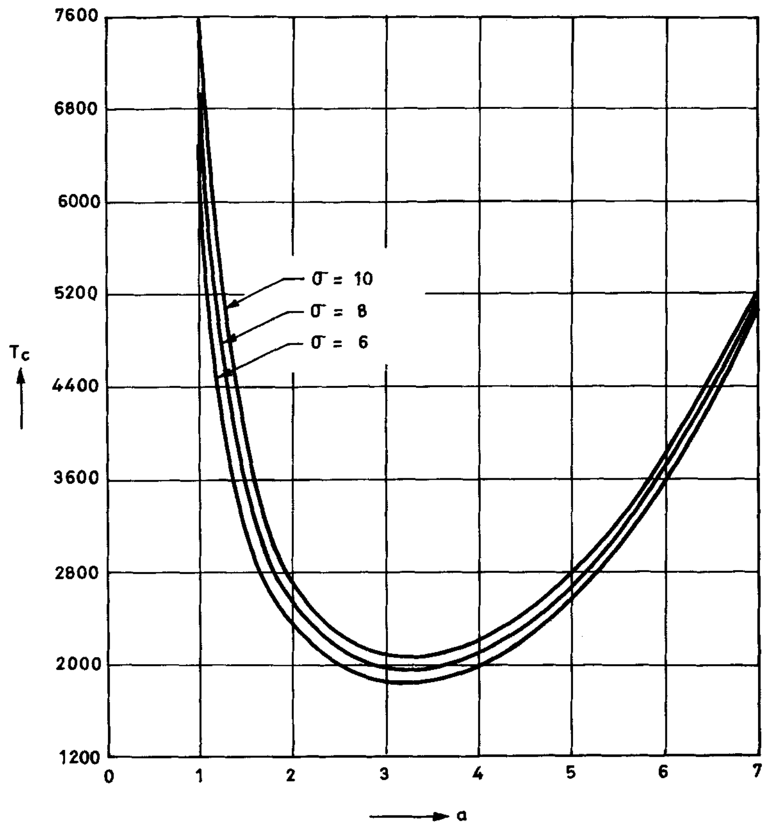

Figure 3. Stability diagram (for $\eta = 0{\cdot}9$)

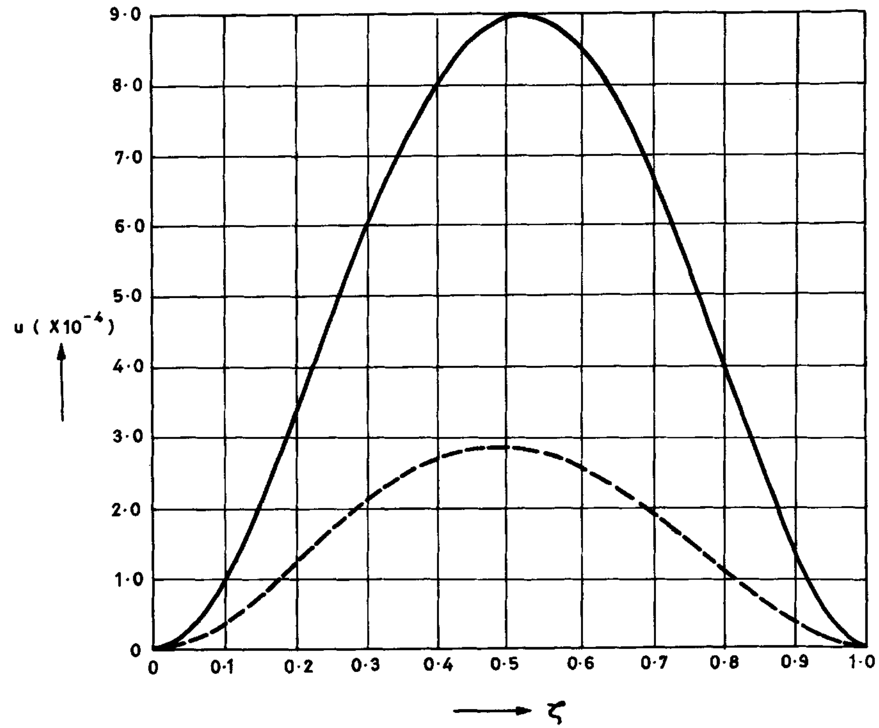

Figure 4. Variation of the eigenfunction u with ζ: ------ Chandrasekhar[1]; —— authors (for $a = 3{\cdot}12$, $\eta = 0{\cdot}9$ and $\sigma = 10$)

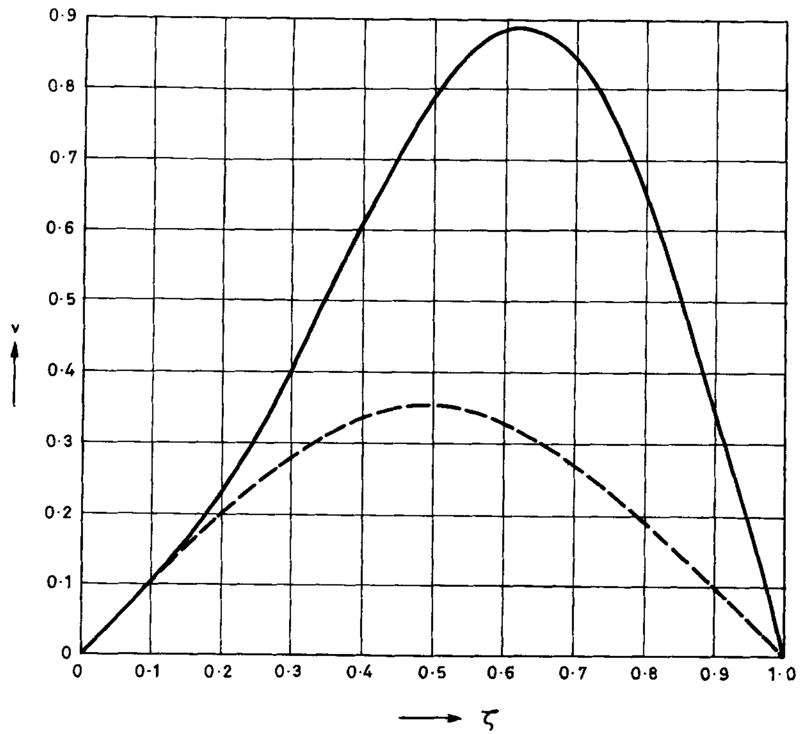

Figure 5. Variation of the eigenfunction v with ζ: ------ Chandrasekhar[1]; —— authors (for $a = 3{\cdot}12$, $\eta = 0{\cdot}9$ and $\sigma = 10$)

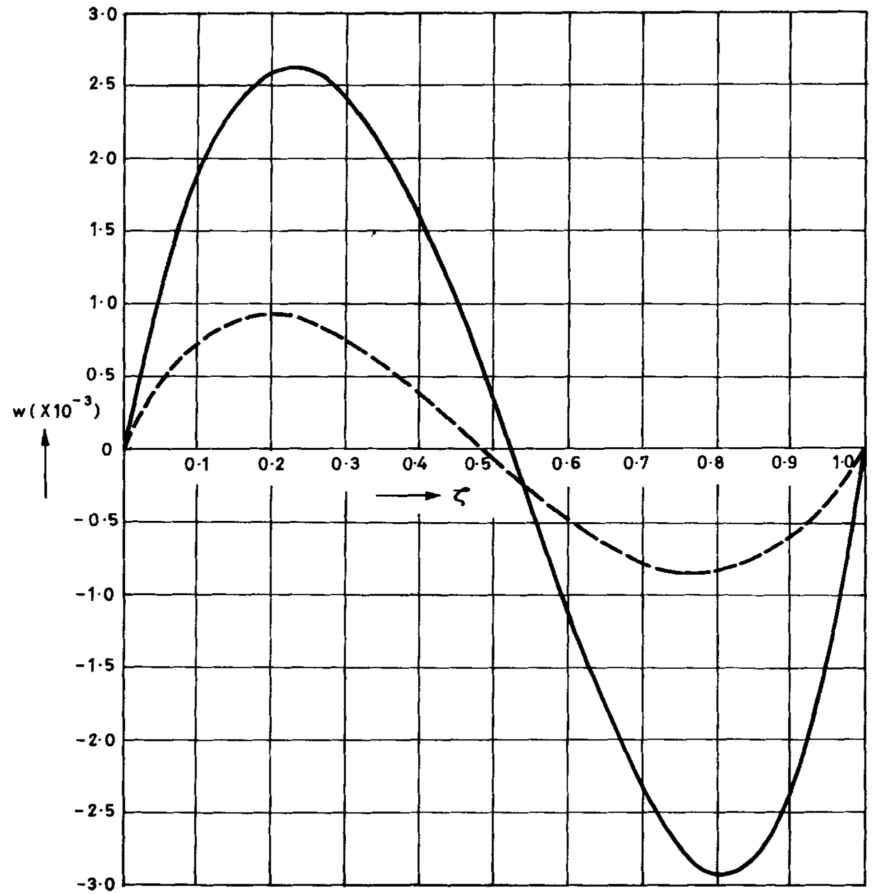

Figure 6. Variation of the eigenfunction w with ζ: ------ Chandrasekhar[1]; —— authors (for $a = 3{\cdot}12$, $\eta = 0{\cdot}9$ and $\sigma = 10$)

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LIST OF SYMBOLS

r, θ, z	Cylindrical co-ordinates
u, v, w	Disturbance velocity components
V	Basic flow velocity
R_1	Radius of the inner cylinder
R_2	Radius of the outer cylinder
$\eta = \frac{R_1}{R_2}$	Ratio of the radii of the cylinders
Ω	Angular velocity
ρ	Density of the fluid
μ	Coefficient of viscosity
$\nu = \frac{\mu}{\rho}$	Kinematic viscosity
P	Pressure
K	Permeability of the porous material
$\sigma = \frac{R_2}{\sqrt{K}}$	Non-dimensional permeability parameter

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